Exercise 5

Use residues to evaluate the definite integrals in Exercises 1 through 7.

$$\int_0^\pi \frac{\cos 2\theta \, d\theta}{1 - 2a\cos\theta + a^2} \quad (-1 < a < 1).$$

$$Ans. \ \frac{a^2\pi}{1-a^2}.$$

Solution

Notice that the integrand is an even function of θ, so the lower limit of integration can be extended to $-\pi$ as long as the integral is divided by 2.

$$\int_0^\pi \frac{\cos 2\theta \, d\theta}{1 - 2a\cos\theta + a^2} = \frac{1}{2}\int_{-\pi}^\pi \frac{\cos 2\theta \, d\theta}{1 - 2a\cos\theta + a^2}$$

Now make the substitution,

$$\begin{aligned} \alpha &= \theta + \pi \quad \rightarrow \quad \theta = \alpha - \pi \\ d\alpha &= d\theta, \end{aligned}$$

so that the integral goes from 0 to 2π.

$$\frac{1}{2}\int_{-\pi}^\pi \frac{\cos 2\theta \, d\theta}{1 - 2a\cos\theta + a^2} = \frac{1}{2}\int_{-\pi+\pi}^{\pi+\pi} \frac{\cos 2(\alpha-\pi) \, d\alpha}{1 - 2a\cos(\alpha-\pi) + a^2} = \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\alpha \, d\alpha}{1 + 2a\cos\alpha + a^2}$$

The integral can now be thought of as one over the unit circle in the complex plane.

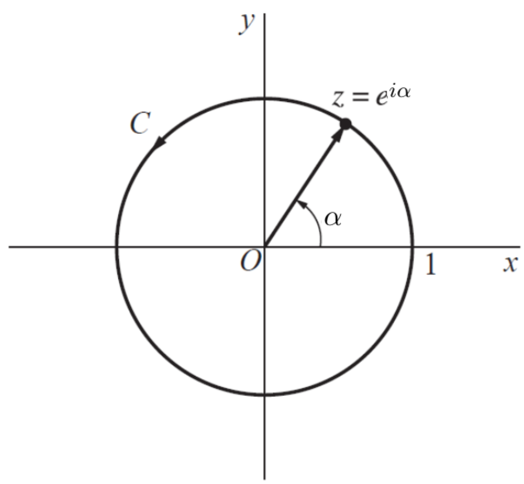

Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

This circle is parameterized in terms of α by $z = e^{i\alpha} = \cos\alpha + i\sin\alpha$. Write $\cos 2\alpha$ and $\cos\alpha$ in terms of z and write $d\alpha$ in terms of dz.

$$\begin{cases} z^2 = e^{2i\alpha} = \cos 2\alpha + i\sin 2\alpha \\ z^{-2} = e^{-2i\alpha} = \cos 2\alpha - i\sin 2\alpha \end{cases} \quad \rightarrow \quad z^2 + z^{-2} = 2\cos 2\alpha \quad \rightarrow \quad \cos 2\alpha = \frac{z^2 + z^{-2}}{2}$$

$$\begin{cases} z = e^{i\alpha} = \cos\alpha + i\sin\alpha \\ z^{-1} = e^{-i\alpha} = \cos\alpha - i\sin\alpha \end{cases} \quad \rightarrow \quad z + z^{-1} = 2\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{z + z^{-1}}{2}$$

$$z = e^{i\alpha} \quad \rightarrow \quad dz = ie^{i\alpha}\,d\alpha = iz\,d\alpha \quad \rightarrow \quad d\alpha = \frac{dz}{iz}$$

With this change of variables the integral in $d\alpha$ will become a positively oriented closed loop integral over the circle's boundary C.

$$\begin{aligned} \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\alpha\,d\alpha}{1 + 2a\cos\alpha + a^2} &= \oint_C \frac{1}{2}\frac{\frac{z^2+z^{-2}}{2}}{1 + 2a\left(\frac{z+z^{-1}}{2}\right) + a^2}\frac{dz}{iz} \\ &= \oint_C \frac{1}{4i}\frac{z^2 + z^{-2}}{az^2 + (a^2+1)z + a}\,dz \\ &= \oint_C \frac{1}{4ai}\frac{z^4 + 1}{z^2\left(z^2 + \frac{a^2+1}{a}z + 1\right)}\,dz \\ &= \oint_C \frac{1}{4ai}\frac{z^4 + 1}{z^2(z+a)\left(z + \frac{1}{a}\right)}\,dz \end{aligned}$$

According to the Cauchy residue theorem, such an integral in the complex plane is equal to $2\pi i$ times the sum of the residues inside C. Determine the singular points of the integrand by solving for the roots of the denominator.

$$4aiz^2(z+a)\left(z + \frac{1}{a}\right) = 0 \quad \rightarrow \quad \begin{cases} z_0 = 0 \\ z_1 = -a \\ z_2 = -\dfrac{1}{a} \end{cases}$$

Since $-1 < a < 1$, there are only two singular points inside the unit circle, namely $z = z_0$ and $z = z_1$, so there are only two residues to calculate.

$$\oint_C \frac{1}{4ai}\frac{z^4+1}{z^2(z+a)\left(z+\frac{1}{a}\right)}\,dz = 2\pi i\left[\operatorname*{Res}_{z=z_0} \frac{1}{4ai}\frac{z^4+1}{z^2(z+a)\left(z+\frac{1}{a}\right)} + \operatorname*{Res}_{z=z_1} \frac{1}{4ai}\frac{z^4+1}{z^2(z+a)\left(z+\frac{1}{a}\right)}\right]$$

The multiplicities of the factors, z and $z + a$, are 2 and 1, respectively, so the residues are calculated by

$$\begin{aligned} \operatorname*{Res}_{z=z_0} \frac{1}{4ai}\frac{z^4+1}{z^2(z+a)\left(z+\frac{1}{a}\right)} &= \frac{\phi_0^{(2-1)}(z_0)}{(2-1)!} = \phi_0'(z_0) \\ \operatorname*{Res}_{z=z_1} \frac{1}{4ai}\frac{z^4+1}{z^2(z+a)\left(z+\frac{1}{a}\right)} &= \phi_1(z_1), \end{aligned}$$

where $\phi_0(z)$ and $\phi_1(z)$ are the same function as the integrand without the factors of z and $z - z_1$, respectively.

$$\phi_0(z) = \frac{1}{4ai} \frac{z^4 + 1}{(z + a)\left(z + \frac{1}{a}\right)}$$
$$\phi_1(z) = \frac{1}{4ai} \frac{z^4 + 1}{z^2\left(z + \frac{1}{a}\right)}$$

Rather than taking a derivative of $\phi_0(z)$, an alternative approach to finding the residue at $z = 0$ would be to use long division to divide $z^4 + 1$ by $4aiz^2(z + a)\left(z + \frac{1}{a}\right)$. The residue would then just be the coefficient of $1/z$.

$$\operatorname*{Res}_{z=z_0} \frac{1}{4ai} \frac{z^4 + 1}{z^2(z + a)\left(z + \frac{1}{a}\right)} = \phi_0'(z_0) = \frac{i(a^2 + 1)}{4a^2}$$
$$\operatorname*{Res}_{z=z_1} \frac{1}{4ai} \frac{z^4 + 1}{z^2(z + a)\left(z + \frac{1}{a}\right)} = \frac{1}{4ai} \frac{z_1^4 + 1}{z_1^2\left(z_1 + \frac{1}{a}\right)} = \frac{i(a^4 + 1)}{4a^2(a^2 - 1)}$$

As a result,

$$\oint_C \frac{1}{4ai} \frac{z^4 + 1}{z^2(z + a)\left(z + \frac{1}{a}\right)}\, dz = 2\pi i \left[\frac{i(a^2 + 1)}{4a^2} + \frac{i(a^4 + 1)}{4a^2(a^2 - 1)}\right] = 2\pi i \left[\frac{ia^2}{2(a^2 - 1)}\right] = \frac{a^2\pi}{1 - a^2}.$$

So then

$$\frac{1}{2} \int_0^{2\pi} \frac{\cos 2\alpha\, d\alpha}{1 + 2a\cos\alpha + a^2} = \frac{a^2\pi}{1 - a^2}.$$

Therefore,

$$\int_0^{\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{a^2\pi}{1 - a^2} \quad (-1 < a < 1).$$